Numbering and Measuring, Epistemologically Considered

[Zählen und Messen, erkenntnisstheoretisch betrachtet]

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Although numbering and measuring are the foundations of the most fruitful, surest, and most exact scientific methods that we all know, relatively little work has been done on ther epistemological foundations. On the philosophical side, strict followers of Kant, who adhere to his system as it originally historically developed in the context of the perceptions and knowledge of his time, in any case hold the axiomata of arithmetic as propositions given *a priori* that constrain the transcendental conception of time in the same sense as the axiomata of geometry do space. Given this view, in both cases the question of a further justification and derivation of these propositions was precluded.

I have tried to show, in earlier essays, that the axiomata of geometry are not propositions given *a priori*, that they should instead be confirmed or refuted by experience. I emphasize once more that Kant's view of space, as a transcendental concept, is not thereby abolished; I believe that only one unjustified particular interpretation of his view is eliminated; which interpretation, in any case, has been disastrous for the metaphysical ambitions of his successors.

Now it is clear that just as the empirical theory, which is advanced by me and by others, regards the axiomata of geometry no longer as propositions both unprovable and without need of proof, it must also justify the origins of the axiomata of arithmetic, which are related to the apprehension of time.

The arithmeticians have hitherto placed, at the beginnings of their deductions, the following propositions as axiomata:

Axiom I. If two quantities are exactly alike a third, they are exactly alike between themselves.

Axiom II. *The Associative Law of Addition*, by H. Grassmann's designation:

$$(a + b) + c = a + (b + c) \ .$$

Axiom III. *The Commutative Law of Addition*:

$$a + b = b + a \ .$$

Axiom IV. Like added to like gives like.

Axiom V. Like added to unlike gives unlike.

Mr. Hermann and Robert Grassmann have entered into this investigation[1] further than the other arithmeticians whose works I know, while at the same time pursuing philosophical considerations; and, in the derivation of the arithmetic conclusions, I shall hereïn follow their course. In doing so, they reduce Axiomata II and III to a single proposition, which we shall call "Grassmann's Axiom", namely

$$(a + b) + 1 = a + (b + 1) \ ,$$

from which they derive the two above more general propositions by the so-called $(n + 1)$ proof. The theory of the addition of pure numbers has, as I hope in the following to show, in fact, obtained the right foundation. But to the question of the objective application of arithmetic to physical magnitudes, to the two concepts of *magnitude* and of *equality in magnitude*, whose meaning remains unexplained in the domain of the facts, is added a third, that of *unit*; and at the same time it seems to me an unnecessary limitation on the validity of the propositions found if from the outset the physical magnitudes are treated only as magnitudes composed of units.

Amongst the newer arithmeticians, E. Schroeder[2] has allied himself with the brothers Grassmann, but has gone deeper in some important discussions. While the former arithmeticians used to conceive the ultimate concept of the number as that of a number of objects, they could not quite free themselves from the laws of the behavior of these objects, and they simply took it as the fact that the number of a group of objects is independent of the order in which they are counted. Mr. Schroeder, as far as I have found, is the first who has recognized (*opere citato* p. 14) that a problem is hidden in this. He also, in my opinion, rightly acknowledged that here is a task for psychology, while on the other hand a requirement to identify those empirical properties that must belong to the objects so that they can be counted.

In addition to this, related discussions, especially on the concept of magnitude, are also found in Paul du Bois-Reymond's *allgemeiner Functionentheorie* [*General Function Theory*] (Tübingen, 1882) Th I, Cap. 1 and A. Elsas' treatise "über die Psychophysik" ["On the Psychophysics"] (Marburg, 1886) pp. 49 ff. Both books, however, are concerned with more specific investigations, without discussing the fundamental basis of arithmetic. Both believe that the concept of magnitude can be derived from that of the line, the former in an empirical conception, the latter in a conception of strict Kantianism. My objection to the latter point of view is already mentioned above, and has been discussed by me in earlier writings. Mr. P. du Bois-Reymond ends his investigation with a

[1] Hermann Grassmann, *Die Ausdehnungslehre.* 1. Aufl., Leipzig 1844. Zweite Aufl. 1878. - Robert Grassmann, *Die Formenlehre oder Mathematik.* Stettin. 1872.

[2] *Lehrbuch der Arithmetik und Algebra.* Leipzig, 1873

paradox, according to which two opposing views, both of which are involved in contradictions, are equally possible.

As the latter author is a highly intelligent mathematician, who has sought with particular interest for the deepest fundamentals of his science, the result of his conclusion has all the more encouraged me to expound my own thoughts on the problem.

In order to give a brief account of the position that leads to simple, logical derivations, and the resolution of the contradictions mentioned, let us consider the following: I regard arithmetic, or the theory of pure numbers, as a method based on purely psychological facts, by which the logical application of a system of signs (namely, the numbers) of unlimited extension and unlimited possibility of refinement is taught. Arithmetic examines, in particular, which different combinations of these signs (calculating operations) lead to the same result. amongst st other things, this also teaches extraordinarily complicated calculations to replace even those that cannot be ended in any finite time by simpler ones. Apart from the specimen thus made of the inner folly of our thought, such a procedure would, at first, be a purely insipid play with imagined objects, which P. du Bois-Reymond mockingly compares to the move of a knight on the chessboard, if it did not permit so extraordinarily useful applications. For by means of this system of numbers we give descriptions of the outcomes of real objects that, where applicable, can reach any degree of accuracy required, and by means of which numbers measuring outcomes are calculated for a large number of cases where natural bodies meet or interact, under the rule of known natural laws.

But then we must ask: What is the objective significance of expressing relations of real objects by given numbers as magnitudes, and under what conditions can we do this? This question, as we shall find, is divided into two simpler ones, namely:

1. What is the objective significance of our declaring two objects in some way equal?

2. What nature must the physical connection of two objects have in order that we may associate comparable attributes of these objects as *additive*, and may regard these attributes as *magnitudes* able to be expressed by denominating numbers? Namely, we consider denominating numbers to be constructed from their parts or units by addition.

The Lawful Series of Numbers.

Counting is a process that is based on our ability to retain the order in which a consciousness act has occurred in temporal succession. We can first consider the numbers as a series of arbitrarily chosen signs, for which only a definite kind of succession is held by us to be lawful, or according to ordinary expression "natural". The designation as the "natural" number series has probably simply attended upon one particular application of counting, namely to the determination of the quantity of given real things. In throwing one of these one after the other by number [into a lot], the numbers follow one another in their regular series in a natural process. This has nothing to do with the order of the symbols for numbers; as the signs in different languages are different, their order could also be arbitrarily determined, so long a certain order is then unalterably fixed as the normal or regular. This order is, in fact, a norm or law given by men, our forefathers, who have worked out the language. I emphasize this difference because the alleged "naturalism" of this series of numbers is related to an incomplete analysis of the concept of number. The mathematicians denote this regular series of numbers as those of "the positive integers". This series of numbers is much more strongly committed to our memory than any other series, which commitment undoubtedly rests on its much more frequent repetition. We therefore also use them in order to consolidate the commitment of other orders to our memory; that is that we use these numbers as *ordinal numbers*.

Uniqueness of the Sequence.

In the series of numbers, advancing and reversing are not equivalent but essentially different processes, as the sequence of perceptions in time, whereäs in the case of lines that are continuous in space and without change in time, none of the two possible directions of progress is distinguished from the other.

Actually, in our consciousness, every present act, whether it be perception, feeling, or will, combines with the memories of past acts — and not those of the future, which are not yet present in consciousness — in a manner consciously and specifically different from the memories that exist beside it. In this way, the present conception is located in contrast to the subsequent one, according to the apprehension of time, a relation that cannot be reversed, and to which every idea that enters into our consciousness is necessarily subject. In this sense the ordering into the sequence of time is the inevitable form of our inner intuïtion.

Sense of the Term.

According to previous discussions each number is only determined by its position in the lawful series.

The symbol "one" is applied to the part of the sequence with which we begin.

Two is the number that follows immediately after *one*; that is to say without interposition of another number in the regular series.

Three is the number that follows immediately after *two*; &c.

There is no reason to discontinue this series somewhere, nor in it to return to a previously used symbol. The decimal system makes it possible, in fact, by a combination of ten different digits, to continue the series indefinitely in a simple and easily understood manner, without ever repeating a numeral.[3]

The numbers that follow a given number in the regular series are called

[3]"*Number theory*" examines numbers in which, after a certain number, *one* follows again and again, which therefore periodically recur.

higher, and those that precede them are called *lower*.[4] There is the complete disjunction, which is grounded in the nature of time series, and we can express as:

Axiom VI. *If two numbers are different, one of them must be higher than the other.*

Addition of Pure Numbers.

To make general propositions about the numbers, I use the familiar symbolism of letters. Each lower-case letter of the Latin alphabet shall designate any number, but always the same within each individual theorem or calculation.

Key: When I represent any number with a letter, for example "a", I will represent the subsequent number in the regular series with "(a + 1)".

This symbol "$(a+1)$" thus should here initially have no meaning other than the one just specified. Generally in any case, as usual, the parentheses signify that the numbers enclosed by them are first to be combined into a number before the other prescribed operations are carried out.

The equality sign [as in] "$a = b$" designates in the pure theory of numbers exactly the same as does "a is the same number as b". Hence, from

$$\begin{aligned} a &= b \\ c &= b \end{aligned} \quad ,$$

$a = c$ immediately follows, since the upper two equations state that both a and c are the same number as b. This establishes the validity of Axiom I for the integers in the pure theory of numbers.

Count of the Numbers.

We now consider the normal number range as determined and given. Now we can consider its members themselves as a series of representations given in our consciousness, whose order from any arbitrarily chosen member can be designated by the series of numbers beginning with *one*.

Definition: I refer with "$(a+b)$" to that number of the main sequence at which I arrive when I'm counting *one* at $(a+1)$, *two* at $[(a+1)+1]$, &c, proceeding thus until I reach b.

The description of this process can be summarized by the following equation (H. Grassmann's *Axiom of Addition*):

$$(a+b)+1 = a+(b+1) \ . \tag{1}$$

Explanation: This equation says that, having counted from $(a+1)$ corresponding to *one* onward to b and have found the number denoted by $(a+b)$,

[4] I am avoiding here even *bigger* and *smaller*; this difference is more closely congruent with the concept of cardinal number, about which later.

if I continue counting in that series to $(b+1)$, I come next to the number subsequent to $(a+b)$, namely $[(a+b)+1]$. Thus, I denote $[(a+1)+1]$, with "$[a+(1+1)]$" or with "$(a+2)$", and further denote $[(a+2)+1]$ with "$(a+3)$", and so on without limit.

In the language of arithmetic, we would call this method "*addition*" and the number $(a+b)$ "the *sum*" of a and b, calling a and b "the *summands*"; but I would point out that in the process indicated the variables a and b do not play the same rôle, and so it must be proved exactly that they can be interchanged without changing the sum, which is to be done below. However, if we keep this consideration in mind, we can accept this term, and say that the form of $(a+b)$ dictates that b should be added to a, and $(a+b)$ is the sum of a and b , but the rules are that b follows a, the first of the arguments. a may be called the *first* summand, b *second.* Accordingly in a consistent application of the notation any number $(a+1)$ can be regarded as the sum of the preceding a with the number *one.*

The specified method of addition will always yield the same result in the regular series of numbers for the same numbers a and b. For each of the steps from which we have composed the addition $(a+b)$ is a single step in the series of the positive integers, from $(a+b)$ to $[(a+b)+1]$, and from b to $(b+1)$. Each individual step is executable and, according to our assumptions, each individual step must always have the same successor over the unalterable preservation of the series of numbers in our consciousness.

Thus there will certainly be one number corresponding to the number $(a+b)$, and only one. This theorem would correspond to the content of *Axiom IV* when applied to the pure numbers and to the kind of addition prescribed here.

On the other hand, from the description of the procedure given, it follows that $(a+b)$ is necessarily different from and higher than a, when b is a positive integer.

If c is a number greater than a, I will by a gradual, upward counting necessarily reach c, and will be able to identify the number by which c is counted from a. Let it be the b-th; then

$$c=(a+b)\ .$$

For later citation, let us denote this proposition as "**Axiom VII**": *If a number is c higher than another a then I can represent c as the sum of a with some positive integer b.*

Theorem I: *Of the Order of Execution of Several Additions.* (*The Associative Law* of Grassmann.)

If to a sum $(a+b)$ I should add a number c, then I get the same result if I add $(b+c)$ to the number a. Or, written as as equation,

$$(a+b)+c=a+(b+c) \tag{2}$$

Proof:

The proposition is, according to equation (1), valid for $c = 1$ It will be shown that if it is right for any one value of c, then it is true also for the following $(c + 1)$.

Namely, it is in accordance with equation (1):

$$\begin{aligned}[(a+b)+c]+1 &= (a+b)+(c+1)\\ [a+(b+c)]+1 &= a+[(b+c)+1]\\ &= a+[b+(c+1)]\end{aligned}$$

The latter according to proposition (1).

So if proposition (2) applies to the value of c occurring here, then the left-hand expressions of the first two equations are the same numbers, and therefore also

$$(a+b)+(c+1) = a+[b+(c+1)] \ ;$$

that is that the proposition also applies to $(c + 1)$.

Since, as noted previously, this proposition holds for $c = 1$, it is also true for $c = 2$. Since it holds for $c = 2$, it also holds for $c = 3$, and so forth, without bound.

Corollary: Since the two compound forms in equation (2) have the same meaning, we can introduce for both also the name without parentheses:

$$a+b+c = (a+b)+c = a+(b+c) \tag{2_a}$$

But we must not change the order of a, b, c in these expressions until we have proven the admissibility of such an exchange.

Generalization of the Associative Law.

We first generalize the term given in (2_a).

$$R = a+b+c+d+\ldots+k+1 \tag{2_b}$$

is intended to denote a sum in which the individual additions in the series, as they are written, are carried out; and to abbreviate a description

$$m+R = m+a+b+c+d+\ldots+k+1 \ ,$$

while

$$m+(R) = m+(a+b+c+d+\ldots+k+1) \ ;$$

however, with the meaning of this notation,

$$(R)+m = R+m \ .$$

Other capital Latin letters will be used in the same sense as R.

Then

$$R + b + c + S = [(R) + b + c] + S$$

because here are synonymous expressions. On other hand, according to equation (2_a)

$$(R) + b + c = (R) + (b + c) \ .$$

Thus

$$R + b + c + S = [R + (b + c)] + S = R + (b + c) + S \ ;$$

that is that, instead of adding all the members in order, one can combine two arbitrary middle members into a sum.

Once this happens, the newly formed sum $(b + c)$ is represented by a single number, and you can continue in the same way, and connect any other pair of consecutive numbers, and so on.

Thus, in the case of any number of connections, the order in which the additions prescribed by the individual "+" signs are executed can be changed without changing the total sum.

Theorem II (*The Commutative Law* of H. Grassmann): If, in a sum of two summands, one of the summands is *one*, the order thereof can be interchanged. This corresponds to the equation

$$1 + a = a + 1 \ . \tag{3}$$

Proof: The equation is true for $a = 1$. Again, show that, if it is correct for any definite value of a, it is also for $(a + 1)$. Because, according to equation (1)

$$(1 + a) + 1 = 1 + (a + 1) \ .$$

By assumption, the equation (3) should hold for

$$(1 + a) + 1 = (a + 1) + 1 \ .$$

It follows from these two equations

$$1 + (a + 1) = (a + 1) + 1 \ ,$$

which was to be proved.

Since the proposition is true for $a = 1$, it holds also for $a = 2$, and since it is true for $a = 2$, it is also true for $a = 3$, and so forth, without bound.

Theorem III: *In a sum of two summands, the order of the summands can be changed without changing the number corresponding to the sum.* Written

$$a + b = b + a \ . \tag{4}$$

The theorem is true by Theorem II for $b = 1$ If it is right for a given value of b, it is also true for $(b + 1)$. Because, by Theorem I,

$$(a + b) + 1 = a + (b + 1) \ ,$$

according to our assumption

$$\begin{aligned}(a+b)+1 &= (b+a)+1 = 1+(b+a) \\ &= (1+b)+a = (b+1)+a\end{aligned} .$$

Of the last three steps, the first and the last are made by Theorem II, equation (3); the middle by Theorem I, equation (2). Consequently,

$$a+(b+1)=(b+1)+a$$

which was to be proved.

From the proposition

$$a+1=1+a ,$$

follows thus

$$a+2=2+a ,$$

again from that

$$a+3=3+a ,$$

and so on without bound.

Proof of Axiom V. If a and f are different numbers, we can, as shown in Axiom VII, always find a positive integer b, such that

$$(a+b)=f .$$

Then

$$c+f=c+(a+b)=(c+a)+b .$$

Accordingly, $(c+a)$ is then necessarily different from $(c+f)$, that is that different numbers, added to the same number, give different sums.

However, since by Theorem III

$$\begin{aligned} c+f &= f+c \\ a+c &= c+a \end{aligned} ;$$

thus this last conclusion may also be written

$$(f+c)=(a+c)+b ,$$

that is that *the same number, added to different numbers, gives different sums.*

From this follows the theorem, which is important for the theory of subtraction and equations, that two numbers that, when the same number is added to each of them, give the same sum must be identical.

Interchanging the Order of Any Number of Elements.

In the case of the previously described method of counting for addition, two series of numbers, which had remained in their normal order, were combined in pairs so that $(n+1)$ would be paired with with 1, $(n+2)$ with 2, &c. Only the starting points of both sequences in the number series were different.

We shall now consider the more general case of a coördination of the elements of two series, one of which is supposed to preserve a definite sequence, and can therefore be represented by the numerals, and the other a variable sequence. For the latter, let us use as signs the letters of the Greek alphabet. These letters also have a sequence imprinted on our memory, as given in the usual arrangement of their alphabet; but we will only use them as a series that is distinguished by many other possible accidental aspects, the firmer remembrance of which relieves us of the survey. On the other hand, we require that, in the changes to be made, the sequence of these elements be omitted, and none be repeated. This is the easiest thing to do in our consideration when we determine that the group as elements should contain all the letters that, in the traditional order of the alphabet, are for example before κ.

Conversion of two successive elements of a row.

If two successive numbers n and $(n+1)$ have two elements, for example ε and ζ, assigned to them, n can be connected either to ε or to ζ, which gives two types of assignment,

$$\begin{matrix} n & (n+1) \\ \varepsilon & \zeta \end{matrix}$$

or

$$\begin{matrix} n & (n+1) \\ \zeta & \varepsilon \end{matrix}$$

If we remove the first of these two orders, replacing it with the other, leaving all other pairs assigned by a number and a letter unchanged, no number loses the assigned letter, no letter the assigned number, we do not repeat any letter, nor let any one be lost. Thus, if the series containing the first two pairs mentioned above was a group without gaps and without repetitions before conversion, the result of the conversion is as well.

By a suitable repetition of such transpositions of adjacent elements, I can make any element of the group into the first row in the series without producing repetitions or gaps in the series. For if the selected element ξ is the nth, I can interchange it with the $(n-1)$-th, then the $(n-2)$-th, &c, so that its ordinate is always lower, until I finally reached the lowest number of place in the group, namely 1.

In the same way, I can make every element of the series, the position number of which is higher than m, into the mth member of the group without producing gaps or repetitions. In the latter method, those members of the series, the position number of which is less than m, remain unchanged.

It follows that *by continually exchanging neighboring members of a group, I can produce every possible order of their members without letting elements be omitted and without repeating them.* For I can prescribe arbitrarily which one should be the first member of the series, and make it so by the method described. Then I can prescribe which one is to become the second, and make it so likewise. In this case, the element that has just become the first is not removed from its position. Then I can determine the third &c to the last.

Theorem IV: *Attributes of a series of elements that do not change when any adjacent elements are interchanged with each other in their sequence do not change with any possible change in the order of the elements.*

This leads us to *the generalization of the Commutative Law of Addition.*

The capital letters may again mean, as in the generalization of the Associative Law, sums of any number of numbers. Then by the generalized Associative Law,

$$R + a + b + S = R + (a + b) + S \ .$$

By the Commutative Law for two terms,

$$a + b = b + a \ .$$

So since then $(a + b)$ is the same number as $(b + a)$

$$\begin{aligned} R + a + b + S &= R + (b + a) + S \\ &= R + b + a + S \end{aligned} \ .$$

The latter again according to the Associative Law.

Since in the given sum any two successive elements can be interchanged without changing the amount of the sum, we can, according to Theorem IV, exchange them all together and place them in arbitrary order without changing the sum.

Thus the five basic axiomata of addition are proved and deduced from the concept of addition that we have laid down. It is still to be shown that this concept coïncides with that which proceeds from the determination of the number of objects to be counted.

This leads us first to the concept of the *cardinal number* of elements of a group. If I need the complete number series from 1 to n to assign a number to each element of the group, I call n the "*cardinal number*" of members of the group. The discussion that preceded the establishment of Theorem IV shows that *the cardinal number of terms is kept constant by changes in the order of the terms*, if omissions and repetitions thereöf are avoided.

This theorem is now applicable to real objects, temporarily named with "α", "β", "γ", &c. But these objects must, at least as long as the result of a given count be valid, satisfy certain conditions, so that they may be counted. They must not disappear or merge with others, neither can they be divided into two parts, and no new one is to be added, so that each name given in the form of a Greek letter also corresponds continuously and uniquely to a single, recognizable persistent object. Whether these conditions are met by a given class of objects can of course only be determined by experience.[5]

That *the cardinal number of the members of two groups of which there is no element in common is, according to the preceding concept of addition, equal to the sum of the cardinal numbers of the members of the two individual groups* is given from the counting method described above. In order to find the total number, a single group could be counted. If it has p members, the number $(p + 1)$ would apply to the first member of the other group, $(p + 2)$ to the second, and so on, so that the total number of members of both groups is exactly found by the same process of counting as the sum defined above of the two numbers that indicate the number of elements of each group.

The concept described above of addition is thus consistent, in fact, with the concept of the same that arises out of the determination of the total number across groups of definable objects, but has the advantage that it can be obtained without reference to external experience.

The axiomata of addition, which are necessary for the foundation of arithmetic, are proved for the concept of number and of the sum from which we have proceeded, which is taken only from internal intuïtion, and at the same time the consistency can be established of the result of this kind of addition with that which is derived from the counting of external, measurable objects.

The theories of subtraction and of multiplication develop from here without further difficulties, since the *difference* $(a - b)$ is defined as the number that must be added to b in order to obtain a as the sum, and multiplication as an addition of a total number of identical numbers. Here I need only to refer to Mr. Grassmann who defines the multiplication of pure numbers by the two equations

$$\begin{aligned} 1 \cdot a &= a \, , \\ (b+1) \cdot a &= b \cdot a + a \, . \end{aligned}$$

With respect to subtraction, it need only be noticed that the numbers as signs of an order in the descending direction can be continued into the infinite from 1 backwards to 0, thence to (-1), (-2), &c, and these new signs treated even as the previously used positive integers. Then the difference of two numbers always has one meaning, and only one; it is thus unambiguously determined.

[5] During the printing I learned that Mr. Professor L. Kronecker has developed the concepts of number and of cardinal number in a lecture of the last winter semester.

The agreement, like the difference, between the laws of addition and those multiplication is still to be discussed here.

The Associative Law and the Commutative Law apply to both operations. That is that, as we have seen,

$$\begin{aligned} (a+b)+c &= a+(b+c) \\ a+b &= b+a \end{aligned} .$$

But also

$$\begin{aligned} (a\cdot b)\cdot c &= a\cdot(b\cdot c) \\ a\cdot b &= b\cdot a \end{aligned} .$$

A difference between the fundamental properties of both operations is shown only if a total number n of same numbers a is combined by each of them. Combined by adding, these give the product $n \cdot a$, which is itself subject to the Commutative Law again:

$$n \cdot a = a \cdot n \ .$$

By multiplication of n equal factors, however, you get the power a^n that, except in special cases of a and n, cannot be interchanged one with the other without changing the value of the power.

Similarly, for the relation of each of these operations in their connection with the next higher in the one case, an analogy is shown

$$\begin{aligned} (a+b)\cdot c &= (a\cdot c)+(b\cdot c) \\ (a\cdot b)^c &= (a^c)\cdot(b^c) \end{aligned} .$$

But the analogy is missing for commutation, as it does not have the same further relationship for $c \cdot (a+b)$ on the one hand and $c^{a \cdot b}$ on the other.

Denominated Numbers.

Counting unalike pieces, in the manner that we have discussed above, we generally need only to ascertain their completeness.

Of much greater importance and extended application is the counting of similar objects. Such objects, which are equal and counted in any particular relation, are called the *units* of the numeration, the number of which we denote as a *denominated number*, the particular kind of units that it summarizes being the *denomination of the number*.

A cardinal number, as we have seen above, is decomposable into parts, which may be *additively* summed into a whole. The sum of two denominated numbers of the same denomination is the total number of all their units, and therefore necessarily a denominated number of the same denomination. If we have to compare two different groups of different cardinal numbers, we denote the higher number as the *larger* one, the lower number as the *smaller* one. If both have the same number, we denote them as *equal.*

Objects or attributes of objects that, compared with similar things, allow the distinction into the greater, equal, or smaller, we call *magnitudes.* If we can express them by a denominated number, we call the number the *value* of the magnitude; the method by which we find the given number, the *measurement* of the magnitude. Moreover, in many investigations that have actually been carried out, we are only able to trace the measurement to units arbitrarily chosen or given by the instrument used; then the numbers that we find have only the value of *ratios* until these units are reduced to generally known units (*absolute units of physics*). These generally known units, however, are not to be defined by their concept, but only according to certain natural phenomena (weights, measures) or certain natural processes (day, beat of a pendulum). That they are more generally known by tradition amongst st men, does not alter the business and the concept of measuring, and appears to it only as a contingency.

In what follows, we will have to examine the circumstances under which we express magnitudes by means of denominated numbers, that is to say by means of their values, and what we can achieve with that knowledge.

For this purpose, however, we must first discuss the concept of equality and magnitude according to its objective significance.

Physical Equality.

The special relation that may exist between the attributes of two objects, and that is what we call by the name of "equality" is characterizes by *Axiom I* quoted earlier above: *When two quantities equal a third, they are each equal to the other.* That is at the same time that the relation of equality is a symmetric one. From

$$\begin{array}{rcl} a & = & c \\ b & = & c \end{array},$$

follows $a = b$ as well as $b = a$.

Equality between comparable attributes of two objects is an exceptional case, and will, therefore, appear in actual observation only when two similar objects, under suitable conditions, produce a peculiar outcome, which does not generally occur between other similar pairs of objects.

We shall call the method by which we place the two objects under suitable conditions, in order to observe the outcome mentioned above, and to determine its occurence or non-occurrence, the *method of comparison.*

If this procedure of comparison is to give a reliable account of the equality or difference of a particular attribute of two objects, then its result will depend exclusively on the condition that both objects possess the attribute in question in the given measure, always assuming that the procedure of the comparison is properly executed.

From the above axiom it follows that *the result of the comparison must remain unchanged when the two objects are interchanged.*

It follows that if two objects a and b are equal and that, by earlier observation by means of the same method of comparison, a is equal to a third object c, the corresponding comparison of b and c as equal must now be made. These are demands that we have to make of the relevant method of comparison. *Only such procedures are capable of establishing an equality that satisfies the stated requirements.*

The fact that "equal magnitudes can be substituted each for another" follows, from these assumptions, for the result on whose observation we base the ascertainment of their equality.

But the equality of other effects or relations of the objects in question can also be connected with equality in the case so far discussed, so that in these latter relations the objects in question can also be substituted each for another. We typically express this thought in words to the effect that we treat the ability of the objects to produce a result in the first kind of comparison as itself an attribute of those same objects, of the same magnitude across the objects equal under the comparison, and treat the other effects in which there is equality as effects of that attribute, or, as a result of experience, as dependent on that attribute alone. The meaning of such an assertion is always that objects that have proved equal in comparison of the sort that decides on the equality of this particular attribute can also replace one another in the designated other cases without altering the result.

Magnitudes whose equality or inequality are to be decided by the same method of comparison are called "qualitatively alike". By separating the attribute whose equality or inequality is determined by abstraction from all that is different in the objects, only a difference of magnitude is left for the corresponding attributes of different objects.[6]

I would like to explain the meaning of these abstract sentences in the case of some familiar examples.

Weights. If I place two bodies on the two pans of a true balance, the scales will not be in equilibrium, but a pan will sink.

In exception, I shall find some pair of bodies a and b that, placed on the balance, do not disturb its equilibrium.

If I then swap a with b, the balance must remain in equilibrium. This is the familiar test of whether the balance is true, that is to say of whether the equilibrium in this balance indicates equality of the weights.

Finally it is confirmed that if not only the weight of a is the same as that of b, but also as that of c, then $b = c$. The equilibrium of the weights on a true scale is, in fact, a method of determining equality of a sort.

The bodies, the weight of which we compare, can, by the way, consist of a great variety of substances, of various shapes and volumes. The weight that we equate is only an attribute separated by abstraction. If we designate the bodies themselves as weights, and these weights as magnitudes, we can do so only where we can ignore all the other qualities of these bodies. This has its

[6]Mr. H. Grassmann's determination of equality, "the same thing as can always be said of it, or more generally, which can be mutually substituted in every judgment", would require that, in every single case of equality, this most general demand, which is subject to misunderstanding, be applied.

practical significance, as often as we observe or produce processes, in the course of which only this attribute of the participating bodies comes into consideration, that is to say in which bodies that equilibrate the balance can each replace another. This is, for example, the case when we measure the persistence of the bodies concerned. But that bodies of equal weight also have the same mass and can also be replaced in the latter relation does not follow from the concept of equality, but only from our knowledge of this particular law of nature for this particular relation.

Distances of Two Points.

The simplest geometric structure for which a magnitude may be determined is the distance of a pair of points. But in order to give a definite value to the distance at least for the time at which they were measured, the points must have a fixed connection, for example as two compass tips. The known method of comparing the distance between two pairs is to examine whether they can be brought to congruence or not. Empirically confirming that this method of equivalence is suitable is that the congruence is found in every situation, as the two pairs of points are interchanged again and again in any way, that two pairs of points, which are congruent to a third, are also congruent each to the other. Thus we can form the concept of equal distances or separations.

From there, one can proceed to the concept of the straight line and its length. Think of two fixed points through which the line is to go. A *straight line* is one in which no point can change its position without changing at least one of its distances from the fixed points. A *curved line*, on the other hand, can be rotated about two of its points, the position of the other points changing, but not the distance from the two fixed points. The length of two finite straight lines is equal if their endpoints have the same distance, that is that they are congruent, and the lines are congruent. The notion of length gives something more than the notion of distance. If we consider two pairs of points of different distances a with b and a with c, coïncident in a and in a straight line, so that a piece of this line is common to both, either b falls on the line ac, or c on the line ab. This gives a contrast equivalent to that of the larger and smaller; whereas the concept of the distance directly offers [only] that of the equal or unequal.

The measurement of time presupposes that physical processes have been found that, in the same way and under the same conditions, when they have begun at the same moment, end at the same time, as for example days, pendula, drainage of sand and water clocks. The justification for the assumption of the unchanged duration in the repetition lies only in the circumstance that all the different methods of time measurement, carefully executed, always produce consistent results. If two such processes a and b begin simultaneously and end at the same time, they proceed not only in equal time but in the same time. With regard to time, there is no difference between them, and therefore no change is possible. And if a third process c, beginning with a at the same time, ends in the same time, it also does so with the simultaneous b.

We compare the *brightness* of two visible fields by bringing them together so that any other delimitation between them falls away except for the difference in brightness, and see whether a recognizable boundary remains between them.

We compare *tone pitches*, when differences are small, by the phenomenon of beating, which must be absent if the pitches are the same. We compare the *intensities of electric currents* with a differential galvanometer, which remains at rest when they are equal, &c.

It is therefore necessary, for the task of establishing equality in different relations, to seek very different physical means, which, however, must satisfy all the requirements set forth above, if they are to prove equality.

The first Axiom "If two quantities equal a third, they are equal" is not a law of of objective significance, but determines only which physical relations we may recognize as equal.

In order to cite examples where this Axiom about a third entity is almost a mechanical construction, I note the grinding of flat glass surfaces. If two of these are abraded with one rotation, the two can be spherical, one concave and the other convex. When three are alternately sanded each against the others, they must finally be flat. Likewise, the edges of precise metal rulers are made straight by grinding three of them each against the others.

About Additive Physical Connections[7] of Comparable Magnitudes.

The comparison of magnitudes, as far as we have hitherto treated, gives us information on the question of whether they are equal or unequal, but not yet a measure for the magnitude of their difference, should they be different. If, however, the quantities in question are to be determined completely by means of denominated numbers, the larger of the two numbers must be represented as the sum of the smaller with their difference. It is therefore necessary to examine under which conditions we may express a physical connection of similar magnitudes as addition.

The mode of connection in question is generally dependent on the type of variables to be connected. We add, for example, weights by simply placing them in the same scales. We add periods of time by letting the second begin exactly at the moment when the first ceases; we add lengths by placing them in a certain way, namely, in a *straight line*, &.

1. *Shared Kind of Sum and Summands.* Since the discussed connected quantities are to be of a certain kind, their result cannot be changed if I exchange one or more of the magnitudes with equal magnitudes. For these are replaced by the same number, which they themselves had. But the result of the combination, if it is to represent the sum of the connected magnitudes, must likewise be the parts, since the sum of two denominated numbers is again a number of the same denomination, as already noted above. *Thus, the method of comparison, with which we have ascertained the equality of the parts to be exchanged, must*

[7] "Connection" ["Verknüpfung"] is a Grassmannish term, mostly subjective in his use, but only used objectively here.

be decided on the equality of the result of the combination when the parts are exchanged. This is the actual meaning of the requirement that the sum of magnitudes of the same kind must be the same as that of the summand.

For example, in a sum of weights, replace the individual pieces with those of other material, but equal weight. The sum then retains the same weight; but their other physical attributes may change.

2. *Commutative Law.* The result of the addition is independent of the order in which the summands are connected. The same must apply to physical connections, which should be regarded as additions.

3. *Associative Law.* The combination of two similar magnitudes can also be effected physically, in that, instead of the two, an undivided quantity of the same kind is employed that is equal to its sum. In this way, the two are then additively united before all others.

For example, weighing a five-gram piece instead of five individual gram-pieces.

The result of the connection, therefore, must not be altered by the fact that, instead of some of the magnitudes to be connected, I introduce others equal to the sum of them.

It can be shown, however, that when the first two demands are generally fulfilled, the third is fulfilled.

Let the elements be arranged one after the other in order, as they are to be connected with each other in a first case, so that each one is appended to the result of the conjunction of the preceding one, in the same manner as we have described above for the addition of $[a + b + c + \&c]$. If, in a second case, we are required to link any of these magnitudes before the others, we may place them in the first and second position, according to the Commutative Law, which is supposed to apply, to change the order without changing the result. Then, according to the first of our above conditions, we can also replace the result of this connection by a different undivided object, which is equal in magnitude of the same kind.

After this is done, we can, in turn, bring the two nearest magnitudes or sums of magnitudes to the first two places, and so forth, until all are connected in the prescribed order. In none of these operations do we change the magnitude of the finite result of the operations.

A physical conjunction of magnitudes of the same kind can thus be regarded as addition, if the result of the connection, compared as a quantity of the same kind, is not altered, either by the interchange of the individual elements amongst themselves, or by the interchange of connected elements with like elements of the same magnitudes.

If, in the manner described above, we have found the method of connecting the corresponding magnitudes, we can also find which are larger, which are smaller. The product of the additive combination, the whole, is greater than the parts of which it is made. We have never doubted the simplest things with which we had to deal from the earliest youth, such as times, lengths, and weights, which were greater and smaller, because we had always known additive methods of connecting them. It is, however, to be borne in mind that the method of comparison, as we have described above, generally only tells us whether the magnitudes are equal or unequal. If two quantities x are equal each to the other, then all the functions of the same are likewise equal. Of these some will increase as x increases; others decrease. Which of these functions allow an additive physical connection is to be decided only by special experience. This is the case where two kinds of additive connection are possible. Thus we determine exactly by the same method of comparison whether two wires have the same galvanic resistance ω, and have the same conduction λ, since

$$\omega = \frac{1}{\lambda} \ .$$

We add resistances when we connect the wires one behind the other, so that the electricity that has been passed through must pass through each individual one after the other. The conduction of the wires is added when we place the wires next to each other, and connect all their beginnings to each other, and all their ends together. Thus, as physical quantities, we objectify two different functions of the same variable. If a wire has a higher resistance, it has a lower conductivity and vice versa. The question of which is greater, which is smaller, is therefore answered in the opposite sense. Electrical capacitors (Leyden jars) can also be connected side by side and one after the other. In the former case they add the capacitance; voltage in the latter case. The former grows as the latter decreases.

Again we must not be astonished if the axiomata of addition occur in the course of nature, since we recognize as addition only those physical connections that satisfy the axiomata of addition.

Divisibility of Magnitudes and of Units.

So far, we have not yet further analyzed magnitudes into units. The concept of magnitude, as well as those of its equality and of its addition, could be obtained without such a decomposition. The greatest simplification of the representation of the magnitudes is, however, obtained only when we dissolve them into units and express them as denominated numbers.

Magnitudes that can be added are, in general, also divisible. If every one of the magnitudes may be regarded as additive according to the addition method that is valid for a magnitude of this kind, composed of a total number of equal parts, each one of these magnitudes may, according to the law of association of addition, where only the value of this quantity is significant, be replaced by the sum of its parts. Thus it is replaced by a denominated number, other cardinal numbers of the same sort by other numbers of the same parts. The description of the individual magnitudes of the same kind can then be conveyed, by mere statement of the numbers, to a listener who knows the same parts selected as a unit. If the quantities that occur are not exhausted by means of the chosen units, the units are again divided in a known manner, and in this way a determination of the value of each of the occurring quantities can be given to arbitrary precision. However, perfect accuracy can only be achieved for rational situations.

Irrational situations may occur in the real objects; in figures, however, they can never be represented completely accurately, but their value can be included only between arbitrarily narrowed boundaries. This narrowing between bounds is sufficient for all calculations of such functions whose magnitudes, with decreasing variations of the magnitudes from which they depend, also undergo ever smaller changes, which can ultimately fall below any given finite magnitude. This applies, in particular, to the calculation of all the differentiable functions of irrational quantities. On the other hand, however, discontinuous functions can also be constructed, for the calculation of which the knowledge of the limits that are still so narrow, between which the irrational value lies, is not sufficient. Against this the representation of irrational quantities by our system of numbers is always insufficient. In geometry and in physics, however, we have not yet encountered such types of discontinuity.

Quantifications of Properties. (Physical Constants, Coëfficients).

Besides the quantities discussed so far, which are directly recognizable as such, because they permit additive combinations, there are a number of other expressions that can also be expressed by means of denominated or undenominated numbers, for which no additive combination is yet known. They are found whenever there is a natural connection between additive quantities in processes that are influenced by the peculiarities of some particular substance, or by a particular body, or by the particular manner of initiation of the process in question.

For example, the refraction law of light indicates that a certain ratio exists between the sine of the angle of incidence and that of the refraction angle of the light beam of a certain wavelength that emanates from a vacuum into a given transparent substance. The number that expresses this relation is, however, different for different transparent substances, and is therefore a property of the same, its refractive power. Similar quantities are specific gravity, heat conductivity, electrical conductivity, heat capacity, &c. These are followed by those values (integrative constants of the dynamics), that remain unchanged during the undisturbed sequence of movements of a limited system of bodies.

Physics gradually succeeded in reducing all these values to units composed of the three fundamental units of time, length and mass by multiplication, exponentiation, and their inverse operations.

The difference between these values and the additive quantities is not strictly distinguished in the language of physicists and of mathematicians. The former values are often called magnitudes, since they are expressed by numbers; the term "coëfficient" denotes their physical natures comparatively better. The difference, however, is not essential; for occasionally new discoveries may lead to additive combinations of such coëfficients, whereby they would enter into the series of directly determinable magnitudes. In some respects, the difference is probably the same as that which the metaphysician of old expressed in the contrast between *extensive* and *intensive magnitudes.* P. du Bois-Reymond calls the first *linear quantities*, the latter *nonlinear*.

On the other hand, it follows from the given derivation that we must first have formed additive quantities before we can determine coëfficients. For an equation, which expresses a law of nature, can only give the value of a coëfficient if all the other magnitudes occurring thereïn are already determined as magnitudes. The determination of additive magnitudes must therefore always proceed before the non-additive values can be found.

Addition of Dissimilar Magnitudes.

A major role in physics is played by such objects that, in different methods of comparison, represent simultaneously two or three or even several different magnitudes, all of which are added with the same kind of physical connection of the objects. The first sort is of the very large number of the magnitudes that have orientation in space, which occur in physics, that is to say magnitudes that indicate a certain value and at the same time a definite orientation, but which can be imagined from several components of fixed orientation (two in the plane, three in space). The simplest way is to give a general overview of the relations, if the components, which in the same way are to be connected to the resultant, as is prescribed by the law of the parallelogram of the forces, are chosen parallel to three right-angled coördinate axes. This class includes displacements of a point in space, its velocity, acceleration, and the corresponding force that moves it; also rotational velocities and vanishingly small rotations, current velocities of heavy liquids, electricity and heat, magnetic moments, &c.

In additive compounds, the equidirectional components are added together; these sums can again be summed up into a result. All physical connections of such magnitudes, in which the result depends only on the magnitude and direction of the finite result, can be regarded as based on such additive connections in two dimensions as in the geometric interpretation of the imaginary magnitudes of Gauss; for several dimension, as the addition of vectors of H. Grassmann and of the quaternions in the theory of R. Hamilton. The Commutative Law must be satisfied again; then we can compose infinitely small rotations of a fixed body about two different axes into a resulting rotation, as well as rotational velocities, but not finite rotations, because in such cases it is no longer insignificant whether one rotates about axis a and then about the axis b, or the other way around.

But even in the mixture of colored light a similar relation prevails. Every quantity of colored light can be represented in relation to its sensible impression as the union of three measures of suitably selected primary colors. Mixture of several colors then acts upon the eye as three portions of light of the three primary colors, which are obtained for each single primary color, are combined by adding the corresponding quantities, which are found in all united single colors. This is the basis for the possibility of the geometrical representation of the laws of color-mixing by means of centered constructions, as Newton first proposed.

Multiplication of Denominated Numbers.

A denominated number $(a \cdot x)$, where x is intended to denote the type of units and a a cardinal number of such units, can be multiplied by a pure number n. It simply falls under the above mentioned definition of products as the sum of n equal summands a. Since the sum of like summands is a quantity equal to them, the product $(n \cdot a)$ is a quantity of the same denomination. The Commutative Law applies to this product, in-so-far as

$$n \cdot (a \cdot x) = a \cdot (n \cdot x) \ ,$$

that is that $(n \cdot a)$ can be treated as a pure number, and a new denominated summand $(n \cdot x)$ formed.

Likewise, the law of multiplication of a sum is given immediately,

$$\begin{aligned} (m+n) \cdot (a \cdot x) &= m \cdot (a \cdot x) + n \cdot (a \cdot x) \\ n \cdot (a \cdot x + b \cdot x) &= n \cdot (a \cdot x) + n \cdot (b \cdot x) \end{aligned} \ .$$

The Multiplication of denominated numbers with pure numbers remains entirely within the framework of the definitions and theoremata that are derived above for the multiplication of pure numbers amongst themselves.

It is different with the multiplication of two or more denominated numbers. This has a meaning only in specific cases where special physical connections between the units concerned are possible, which are subject to the three laws of multiplication,

$$\begin{aligned} a \cdot b &= b \cdot a \\ a \cdot (b \cdot c) &= (a \cdot b) \cdot c \\ a \cdot (b + c) &= a \cdot b + a \cdot c \end{aligned} \ .$$

The best known examples of such multiplicative combinations from geometry are the values of the areas of parallelograms and of the volumes of parallelepipeds, expressed as the product of two or three lengths, namely that of a length with one or two heights. But physics forms a large number of such products of different units, and correspondingly also examples of quotients, powers, and roots. If we denote a length l, a time t, with a mass m, then some denominations are

area	l^2
volume	l^3
velocity	$\frac{l}{t}$
motive force	$\frac{m \cdot l}{t^2}$
work	$\frac{m \cdot l^2}{t^2}$
pressure on a surface	$\frac{m}{l \cdot t^2}$
surface tension	$\frac{m}{t^2}$
density	$\frac{m}{l^3}$
magnetism	$\frac{1}{t} \cdot \sqrt{m \cdot l}$
magnetic force	$\frac{1}{t} \cdot \sqrt{\frac{m}{l}}$

&c.

Most of these combinations are based on the determination of coëfficients; but many of these quantities can also provide additive physical connections such as velocities, currents, forces, pressures, imperviousness, &c.

All these multiplicatively defined units, however, are unequal to those from which they are produced, and acquire significance only through the knowledge of particular geometrical or physical laws.

It is worth mentioning here the special variety of multiplication which H. Grassmann has constructed for vectors in his theory of extension, and which is also basic in the theory of quaternions. This represents another commutative law, namely,

$$a \cdot b = -b \cdot a$$

and gives in fact a great simplification in the description, if not in the calculation, of the values that are produced by the interaction of different vectors.

The product of two vectors is, in this type of calculation, the area of a parallelogram that has both sides; but the parallelogrammatic surface is regarded as positive on the one side, and negative on the other. Seeing one side of the surface, I have to rotate to the right in rotating from side a through the angle with side b; looking at the other side, I rotate left in rotating from b to a. This is the basis of the difference in the sequence $(b \cdot a)$ and $(a \cdot b)$.

It is sufficient here to have mentioned these forms of calculation, and to have described their relation to the formulæ of the pure theory of numbers, since the task of the present work requires only demonstration of the importance and justification of the calculation with pure numbers, and the possibility of its application to physical magnitudes.

That we represent any physical relation as a magnitude can therefore always be based only on the empirical knowledge of certain aspects of its physical behavior when it comes together and interacts with others. I consider the congruence of two spatial magnitudes, which occur in bodies, or are delimited by bodies, in the sense of my earlier work on the axiomata of geometry, as a physical relation that must be empirically determined. *First*, we must know the method of comparing the variables in question, by which their nature is characterized and, *secondly*, either the methods of the additive relation or the laws of nature in which they occur as coëfficients in order to express them by means of denominated numbers.

The great simplification and clarity of conception, which we reach by reducing to quantities the colorful variety of the things and changes with which we are presented, is deeply founded in our formulation of concepts. When we form the concept of a class, we summarize in it everything that is the same in the objects belonging to the class. If we conceive of a physical relation as a denominated number, we have also removed from the concept of their units everything that is attached to them as distinct in reality. They are objects that we consider only as examples of their class, and whose effectiveness, under the method of investigation, depends only on the fact that they are such specimens. In the magnitudes formed from them, only the most incidental of distinctions, that of their cardinal numbers, remains.